Cosmological gravitons and the expansion dynamics during the matter age

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Extending previous results [Phys. Rev. D **56**, 6351 (1997)], we estimate the back reaction of cosmological gravitons in expansion dynamics, during the matter age. Tensor perturbations with scales larger than the Hubble radius are created due to the inequivalence of vacua at different times. During noninflationary phases these perturbations become effective gravitational waves as they enter the Hubble radius, adding new contributions to the energy density of the subhorizon waves. During the radiation epoch the creation of these effective gravitons may lead to a departure from the standard behavior $a(t)\propto t^{1/2}$, with possible consequences to several cosmic processes, such as primordial nucleosynthesis. We examine the implications of this phenomenon during the matter-dominated era, assuming an initial inflationary period. The dynamical equation obeyed by the scale factor is derived and numerically solved for different values of the relevant parameters involved. [S0556-2821(99)08420-9]

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I. INTRODUCTION

In a recent paper [1], we have studied the back reaction of cosmological gravitons on the cosmic dynamics during the radiation epoch. The analysis was based on the following reasoning: the inequivalence of vacuum states at different moments of time leads to the production of tensor perturbations in scales larger than the Hubble length [1–10]. During noninflationary periods of expansion, these very long tensor perturbations (VLTP's) become effective gravitational waves (EGW's) as they enter the Hubble radius, thus adding new contributions to the energy density associated with the subhorizon waves ρ_g [1–5,11,12]. Such a phenomenon can be studied as a process of the ''creation of effective gravitons'' by using the macroscopic formalism to matter creation based on the thermodynamics of open systems [13,14]. In order to deal with the process, a creation pressure term is introduced in the continuity equation obeyed by ρ_g. A dynamical equation for the scale factor $a(t)$ is then derived which takes into account the effective graviton back reaction. A comprehensive explanation of this approach can be found in Ref. [1], where the equation for $a(t)$ was numerically solved for a model in which the universe evolves from an arbitrary initial phase to a radiation-dominated era. In that reference, it was found that, if the barotropic index γ of the equation of state in the first epoch is close to 2/3, the back reaction of the effective gravitational waves makes $a(t)$ deviate appreciably from the standard behavior $a(t)\propto t^{1/2}$.

In the present paper we extend our analysis to the matter-dominated period of the cosmic expansion. In particular, we aim to check a conjecture stated by Sahni [12], according to which the process mentioned above would ultimately lead the universe to expand linearly with time $[a(t)\propto t]$, since the tensor perturbations would enter the Hubble radius during noninflationary periods of expansion (increasing ρ_g) and leave the Hubble radius during inflationary periods (decreasing ρ_g). In Ref. [1] we have shown that the scale factor would turn from $a(t)\propto t^{1/2}$ to $a(t)\propto t$ only if $\gamma\to 2/3$. Several models with the universe evolving linearly with time during its early phases have been proposed, most of them related to string-motivated cosmologies [15,16]. A universe evolving linearly with time during most of its history would lead to important observational consequences, such as those related to the age of the universe, to the luminosity-distance – redshift and the angular-diameter-distance–redshift relations, and to the galaxy number count as a function of the redshift [17,18] (see also [19]).

We should remark that, in order for the mechanism of graviton creation to take place, no ''exotic'' physics has to be assumed. It only requires the validity of quantum mechanics and of general relativity [20] or other related gravity theories, such as scalar-tensor theories [4,21]. Therefore, the study of the possible influence of the cosmological gravitational waves in the expansion dynamics has an importance on its own, a fact that has not been fully appreciated in the literature. By ''mechanism of graviton creation,'' we mean here the quantum mechanical mechanism of the generation of tensor perturbations, due to the inequivalence of vacuum states at different moments of time. Those are the ones referred to at the beggining of the first paragraph and include the perturbations created in scales larger than the Hubble radius, which we have named ''very long tensor perturbations.'' This mechanism should not be confused with the latter transformation of the VLTP's into ''effective gravitational waves,'' during noninflationary eras, which may be regarded as a ''creation of effective gravitons.'' In order to avoid any further confusion, whenever we speak of ''graviton creation,'' we will be referring to the former mechanism, whereas the latter will be identified by the expression ''creation of effective gravitons.''

Another source of confusion may be the expression ''matter creation'' referred to in the first paragraph. We do not deal, in the present paper, with scenarios where *matter* creation (as opposed to *graviton* creation) occurs. What we do is to use the macroscopic formalism of matter creation based

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on the thermodynamics of open systems [13,14] to model the transformation of VLTP's into EGW's. As the very long gravity wave perturbations enter the Hubble radius, they add new contributions to the energy density of the subhorizon waves. This mimics the creation of what we have called effective gravitons. This ''creation'' is described using the formalism mentioned above. (For self-contained descriptions of both the macroscopic formalism of matter creation and of the quantum mechanism of graviton creation see Secs. II and III of Ref. [1], respectively. See also Sec. II below for a short description of the latter mechanism.)

It is also important to realize that inflation is *not* a necessary condition for graviton creation to occurr, although it is a sufficient one. In fact, there seems to exist a widespread misconception in this respect, which is continuously reinforced in the literature by expressions such as ''gravitational waves from inflation,'' ''inflation generated gravitational waves,'' and so on. Note that the early work by Grishchuk on the mechanism of superadiabatic amplification of gravitational waves was first published in 1975 [22], well before the idea of inflation was put forward by Guth [23]. Grishchuk has also addressed this point in Ref. [20]. The gravitational field of the expanding universe acts as a ''pump'' field that supplies energy to the zero-point quantum fluctuations [20]. This a dynamical, not a kinematical, effect and is related to the lack of conformal invariance of the gravitational wave equation [22,24]. It does not happen, for instance, for photons, even in inflationary eras, since the electromagnetic wave equation *is* conformally invariant (unless we relax the assumption of minimal coupling between the gravitational and the electromagnetic field [25–28]). The conditions for the occurrence of superadiabatic amplification has been explained in detail by Tavares and Maia in Ref. [24]. As for its quantum counterpart — graviton creation — see, for instance, the early paper by Ford and Parker [10]. A careful analysis of the formalism used by Allen [2], Abbott and Harari [6], and Allen and Koranda [7] will convince the reader that there is nothing intrinsically ''inflationary'' in the mechanism of graviton creation, in spite of the fact that these three last references deal with inflationary scenarios. This point is made explicit by Maia [3], Maia and Barrow [4], Maia and Lima [5], Tavares and Maia [24], and Maia, Carvalho, and Alcaniz [1]. In particular, there is graviton creation due to the transition from a radiation-dominated to a matter-dominated era (see, for example, [2,7,12]). Nevertheless, the existence of an early inflationary epoch remains the best motivation to study such perturbations. Accordingly, in the present paper, we will assume that the universe went through an accelerated phase during its early stages.

The paper is organized as follows: In Sec. II we present the basic equations related to the spectrum of cosmological gravitational waves. In Sec. III we derive the dynamical equations that govern the expansion dynamics during the matter age, taking into account the graviton back reaction. In Sec. IV we show the numerical solutions of these equations and present our conclusions.

The system of units used is such that $\hbar = c = k_B = 1$.

II. BASIC EQUATIONS

We will consider a homogeneous and isotropic universe, for which the background line element takes the Friedmann-Robertson-Walker (FRW) form

$$ds^2 = dt^2 - a^2(t)dl^2 = a^2(\eta)(d\eta^2 - dl^2), \tag{1}$$

where t and η are, respectively, the cosmic and conformal times, related by

$$dt = a\, d\eta. \tag{2}$$

We will restrict ourselves to the spatially flat case. If the pressure p and energy density ρ of the cosmic fluid are related by the equation of state

$$p = (\gamma - 1)\rho, \tag{3}$$

then the scale factor is [3]

$$a(t) = a_0 \left[1 + \frac{3\gamma H_0}{2}(t - t_0) \right]^{2/(3\gamma)}, \tag{4}$$

where $H_0 \equiv H(t_0)$ and $H \equiv \dot{a}/a$.

We will further assume that the universe evolves from an initial arbitrary era ($\gamma = \gamma$) to a radiation-dominated phase ($\gamma = 4/3$) and then to a matter-dominated period ($\gamma = 1$).

As the inequivalence of vacua appears at different instants of time, we will focus on the time-dependent amplitude of the tensor perturbations $\mu(k,\eta)$, which obeys [1,3,4]

$$\mu''(k,\eta) + \left(k^2 - \frac{a''}{a} \right) \mu(k,\eta) = 0. \tag{5}$$

In the above equation the primes indicate derivatives with respect to the conformal time and k is the comoving wave number, related to the physical wavelength λ and frequency ω by

$$k = \frac{2\pi a}{\lambda} = \omega a. \tag{6}$$

The spectrum of the EGW's can be described by the quantity $P_g(\omega)$, defined in such a way that $P_g(\omega)\, d\omega$ represents the energy per unit volume between the frequencies ω and $\omega + d\omega$ [2–4]. Note that this quantity can be defined only for the EGW's [2,11], i.e., for those perturbations such that

$$\lambda \leq \lambda_H \equiv H^{-1}. \tag{7}$$

In the units used in this paper and assuming an initial vacuum state, $P_g(\omega)$ is given by [2–4]

$$P_g(\omega) = \frac{\omega^3}{\pi^2} \langle N(\omega) \rangle. \tag{8}$$

In the above equation $\langle N(\omega)\rangle$ is the expectation number of the gravitons with frequency ω which is found to be [1–4]

$$\langle N(\omega) \rangle = |\beta(k)|^2, \tag{9}$$

where $\beta(k)$ and $\alpha(k)$ (see below) are the Bogoliubov coeficients relating the creation and annihilation operators (which define the particle states) of different eras. Note that a vacuum state at an early epoch may appear as a multiparticle state at late epochs. This is the essence of the quantum mechanism of particle creation (graviton creation in the present case), and inflation is not a necessary condition for this to occur [1–10].

To see this more clearly, let η_r denote the time at which the universe suffers a transition from a phase where the barotropic parameter is γ_{r-1} to a phase where this parameter is γ_r and let $\mu_{(r)}(\eta)$ be the solution of Eq. (5) for $\eta_r \leq \eta \leq \eta_{r+1}$, representing the properly normalized adiabatic vacuum state [8,9] in this time interval. Then the modes

$$\mathcal{Y}_{(r-1)}(k,\eta)=\frac{\mu_{(r-1)}(k,\eta)}{a_{(r-1)}(\eta)} \quad (\eta<\eta_r)$$
$$=\frac{1}{a_{(r)}(\eta)}[\alpha_r(k)\,\mu_{(r)}(k,\eta)+\beta_r(k)\,\mu^*_{(r)}(k,\eta)] \quad (\eta>\eta_r) \qquad (10)$$

and

$$\mathcal{Y}_{(r)}(k,\eta)=\frac{1}{a_{(r-1)}(\eta)}[\alpha^*_r(k)\,\mu_{(r-1)}(k,\eta)-\beta_r(k)\,\mu^*_{(r-1)}(k,\eta)] \quad (\eta<\eta_r)$$
$$=\frac{\mu_{(r)}(k,\eta)}{a_{(r)}(\eta)} \quad (\eta>\eta_r) \qquad (11)$$

define two quantizations of the field associated with two different Fock spaces. (An asterisk indicates the complex conjugate of a quantity.) It is important to bear in mind that the associated operators in each case represent physical particle observables only inside their respective eras. Moreover,

$$\mathcal{Y}_{(r-1)}(k,\eta)=\alpha_r(k)\,\mathcal{Y}_{(r)}(k,\eta)+\beta_r(k)\,\mathcal{Y}^*_{(r)}(k,\eta). \qquad (12)$$

The annihilation and creation operators of the two quantizations are related by a Bogoliubov transformation:

$$A_{(r)}(\tilde{k})=\alpha_r(k)\,A_{(r-1)}(\tilde{k})+\beta^*_r(k)\,A^\dagger_{(r-1)}(-\tilde{k}), \qquad (13)$$

$$A_{(r-1)}(\tilde{k})=\alpha^*_r(k)\,A_{(r)}(\tilde{k})-\beta^*_r(k)\,A^\dagger_{(r)}(-\tilde{k}). \qquad (14)$$

[The symbol $\tilde{k}$ indicates $(K;\vec{k})$ and $-\tilde{k}$ stands for $(K;-\vec{k})$, where K represents one of the two possible polarization states of the gravitational waves (GW's).]

As a result of Eqs. (13) and (14), the vacuum state at the region $(r-1)$, labeled $|0_{(r-1)}\rangle$, is annihilated by $A_{(r-1)}$ but not by $A_{(r)}$. If $N_{(r)}(\tilde{k})$ is the number operator for the mode $\tilde{k}$ at stage (r),

$$N_{(r)}(\tilde{k})\equiv A^\dagger_{(r)}(\tilde{k})\,A_{(r)}(\tilde{k}), \qquad (15)$$

then [4,8,9]

$$\langle N(\tilde{k})\rangle_r \equiv \langle 0_{(r-1)}|N_{(r)}(\tilde{k})|0_{(r-1)}\rangle=|\beta_r(k)|^2. \qquad (16)$$

Therefore, the vacuum state $|0_{(r-1)}\rangle$ is a multiparticle state when we use the definition of particles appropriate for the epoch $\eta>\eta_r$.

If we impose the continuity of $\mathcal{Y}_{(r-1)}$ and $\mathcal{Y}_{(r)}$ and of their first derivatives at η_r, we find the Bogoliubov coefficients to be [3,4]

$$\alpha_r(k)=i\,[\mu^*_{(r)}(k,\eta_r)\,\mu'_{(r-1)}(k,\eta_r)-\mu_{(r-1)}(k,\eta_r)\,\mu^{*\prime}_{(r)}(k,\eta_r)], \qquad (17)$$

$$\beta_r(k)=i\,[\mu_{(r-1)}(k,\eta_r)\,\mu'_{(r)}(k,\eta_r)-\mu_{(r)}(k,\eta_r)\,\mu'_{(r-1)}(k,\eta_r)]. \qquad (18)$$

(See [1,3,4] for details.)

If the scale factor is given by Eq. (4), then the solution of Eq. (5), representing a properly normalized adiabatic vacuum state, is [1,3,4]

$$\mu_{(r)}(k,\eta)=\frac{\sqrt{\pi}}{2}e^{\theta_r}k^{-1/2}x^{1/2}_{(r)}(\eta)\,H^{(2)}_{m_r}[x_{(r)}(\eta)], \qquad (19)$$

where θ_r is an arbitrary constant phase (whose value is irrelevant for the evaluation of the Bogoliubov coefficients), $H^{(2)}_{m_r}$ is the Hankel function of second kind and order m_r [29], and

$$x_{(r)}(\eta)\equiv k(\eta-\overline{\eta_r}), \qquad (20)$$

$$m_r\equiv q_r-\frac{1}{2}, \qquad (21)$$

$$q_r\equiv\frac{2}{3\gamma_r-2}. \qquad (22)$$

Similar results apply to the solution $\mu_{(r-1)}(k,\eta)$. (Although the above solutions do not apply for $\gamma=2/3$, a possible way for defining adiabatic mode solutions in this case can be found in [30].)

We will denote by α_1, β_1 the coeficients associated with the first transition from an arbitrary phase to the radiation-dominated era, and by α_2, β_2 those related with the transition from the radiation to the matter-dominated epoch. The coefficient β is then given by [2]

$$\beta=\beta_2(k)\alpha_1(k)+\alpha^*_2(k)\beta_1(k). \qquad (23)$$

The total energy density associated with the tensor perturbations is obtained by integrating $P_g(\omega)$,

$$\rho_g(t)=\int_{\omega_{min}(t)}^{\infty}P_g(\omega)\,d\omega, \qquad (24)$$

where

$$\omega_{min}(t)=2\pi H(t). \qquad (25)$$

This infrared cutoff is obviously related to the condition (7) and, since it is time dependent, it is the origin of the process of the creation of EGW's. In a FRW scenario, it will be responsible for the fact that ρ_g does not decay with a^{-4}, as one could expect for a massless particle such as the graviton.

The integral in Eq. (24) is usually written with a finite upper limit. This is done because the above method for obtaining the spectrum, based on the sudden transition approximation between cosmic eras, does not give accurate results for high frequency waves. In fact, it is reliable only for those waves whose periods are much greater than the transitions time scales [4,24]. However, because of the adiabatic theorem [9], the number of created particles should decrease exponentially for the high frequency modes. Hence, an ultraviolet cutoff is imposed and the error produced by doing so is supposed to be small. Our present results will not be affected by this approximation since we are obviously interested only in the very long modes.

III. DYNAMICAL EQUATIONS DURING THE MATTER ERA

Following Ref. [1], we will suppose that, prior to the time t_1, the dominant material content of the universe has an equation of state of the form (3). This stage can be either an inflationary or a noninflationary one. At t_1 a transition occurs, so that the new barotropic parameter is 4/3. As a result of the transition, tensor perturbations are created [1,3,4]. Some of these have wavelengths less than $H^{-1}(t_1)$ (EGW's), but most are created with scales larger than $H^{-1}(t_1)$ (VLTP's). A similar phenomenon will occur at a time t_2 when the universe becomes matter dominated ($\gamma=1$). We will further assume that during this matter-dominated period and until a time, say, t_g, the energy density ρ_g associated with the EGW's is negligible compared with the energy density of matter ρ_m. There is no restriction over the size of the interval t_g-t_2, which can be taken to be arbitrarily small. However, after t_g, the ongoing transformation of VLTP's into EGW's makes the continuous increase of ρ_g start perturbing the dynamics. Thereafter, the universe can be supposed to be filled by two fluids: dust and the one composed by the effective gravitons. The *total* energy density is then $\rho_m+\rho_g$. As gravitons require extremely high energies to interact with matter [4,31], the two fluids can be safely considered to be noninteracting. The effective gravitons behave as a perfect fluid with equation of state [22,32]

$$p_g=\frac{\rho_g}{3}. \tag{26}$$

Nevertheless, it is possible to associate a creation pressure term Π_g with the creation process of these gravitons [1]. The field equations are then written as

$$8\pi G\,(\rho_m+\rho_g)=3\frac{\dot{a}^2}{a^2}, \tag{27}$$

$$8\pi G\,(p_m+p_g+\Pi_g)=-2\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}, \tag{28}$$

$$\dot{n}_g+3Hn_g=\Psi_g, \tag{29}$$

where the overdots indicate derivatives with respect to the cosmic time, $p_m=0$ is the pressure of the dust fluid, and n_g and Ψ_g are the number density and creation rate of the effective gravitons, respectively. This last equation is the novel aspect introduced by the phenomenological formalism to particle (in this case, *effective* gravitons) creation [13,14]. The conservation of the energy-momentum tensor leads to

$$\dot{\rho}+3H(\rho+p+\Pi_g)=0, \tag{30}$$

where ρ is the total energy density ($\rho=\rho_m+\rho_g$) and p is the total pressure ($p=p_m+p_g$, $p_m=0$).

As the two fluids are noninteracting, this conservation equation can be split into

$$\dot{\rho}_m+3H\rho_m=0 \tag{31}$$

and

$$\dot{\rho}_g+3H\left(\frac{4}{3}\rho_g+\Pi_g\right)=0. \tag{32}$$

From Eqs. (27), (28), (31), and (32) we obtain [1]

$$\frac{\ddot{a}}{a}+\frac{1}{2}\frac{\dot{a}^2}{a^2}=4\pi G\left(\rho_g+\frac{\dot{\rho}_g}{3H}\right), \tag{33}$$

$$\frac{\dot{a}^2}{a^2}=\frac{8\pi G}{3}(\rho_m+\rho_g). \tag{34}$$

It is important to mention that, in the method described in [1,3,4], the Bogoliubov coefficients that appear in the calculus of $\langle N(\omega)\rangle$ are evaluated at the transition times t_1 and t_2. Hence, $\rho_g(t)$ is univocally determined in terms of $a(t)$ and $\dot{a}(t)$, that is [1],

$$\rho_g(t)=\rho_g(a(t),\ \dot{a}(t)). \tag{35}$$

Therefore, the system of coupled equations (31), (34), and (35) allows us to obtain $a(t)$, taking into account the back reaction induced by the transformation of very long tensor perturbations into effective gravitational waves.

Let us define

$$A(t)\equiv\frac{a(t)}{a_2} \tag{36}$$

and

$$a_2\equiv a(t_2), \tag{37}$$

$$a_1\equiv a(t_1), \tag{38}$$

$$H_1\equiv H(t_1), \tag{39}$$

$$\sigma \equiv \frac{a_2}{a_1}, \tag{40}$$

Then Eqs. (8)–(24),(31),(34), and the expressions derived for the Bogoliubov coeficients in [3,4] lead to a lengthy set of equations that enable us to evaluate $a(t)$ during the matter age. These equations take the form

$$\dot{A}^2 = \frac{H_1^2}{\sigma^4}\frac{1}{A}\left[1+\kappa\left(\frac{H_1}{m_{Pl}}\right)^2\frac{1}{A}\ G(t)\right], \tag{41}$$

where m_{Pl} is the Planck mass. The values of the constant κ and of the function $G(t)$ depend whether $\gamma=0$ (the initial era is a de Sitter one) or $\gamma\neq 0$ (we will restrict our analysis to inflationary models, so that $\gamma<2/3$).

For $\gamma=0$,

$$\kappa = \frac{2}{3\pi}, \tag{42}$$

$$G(t) = \left(1+\frac{1}{16\sigma^4}\right)\ln\tau + \frac{b_2^4}{2048\pi^4}(\tau^4-1) + \frac{b_2^2}{32\pi^2}j(t) + \ln\left(\frac{b_2\sigma}{b_1}\right) + c_0, \tag{43}$$

$$\tau \equiv \frac{H_1}{b_2\,\sigma^2\dot{A}}, \tag{44}$$

$$c_0 \equiv 16\pi^4\left(\frac{1}{b_1^4} - \frac{1}{b_2^4\,\sigma^4}\right), \tag{45}$$

$$j(t) \equiv N_1(\sigma,\tau)\sin\theta_1 + N_2(\sigma,\tau)\cos\theta_1 + N_3(\sigma)\cos\theta_2 + N_4(\sigma)\sin\theta_2 + N_5(\sigma)J(t), \tag{46}$$

$$\theta_1(t) \equiv 4\pi\sigma(\sigma-1)\frac{\dot{A}}{H_1}, \tag{47}$$

$$\theta_2 \equiv \frac{4\pi(\sigma-1)}{b_2\,\sigma}, \tag{48}$$

$$J(t) \equiv \int_{\theta_1(t)}^{\theta_2}\frac{\cos\theta}{\theta}d\theta, \tag{49}$$

$$N_1(\sigma,\tau) \equiv \frac{b_2}{4\pi}\left[\frac{(3\sigma+1)}{\sigma}\tau^3 - \frac{b_2(11\sigma^3+11\sigma^2-7\sigma+1)}{\pi}\tau\right], \tag{50}$$

$$N_2(\sigma,\tau) \equiv \frac{1}{2}\left[\frac{(11\sigma^2+6\sigma-1)}{\sigma^2}\tau^2 - \frac{b_2^2}{8\pi^2}\tau^4\right], \tag{51}$$

$$N_3(\sigma) \equiv \frac{1}{2}\left[\frac{b_2^2}{8\pi^2} - \frac{(11\sigma^2+6\sigma-1)}{\sigma^2}\right], \tag{52}$$

$$N_4(\sigma) \equiv \frac{b_2}{4\pi}\left[\frac{b_2(11\sigma^3+11\sigma^2-7\sigma+1)}{\pi} - \frac{(3\sigma+1)}{\sigma}\right], \tag{53}$$

$$N_5(\sigma) \equiv -\frac{b_2(\sigma-1)}{\pi\sigma}(11\sigma^3+11\sigma^2-7\sigma+1). \tag{54}$$

In the above equations the free parameters b_1 and b_2 are related to the transition time scales Δt_1 and Δt_2 that are assumed to take place at t_1 and t_2, respectively:

$$\Delta t_1 = \frac{b_1}{H(t_1)} = \frac{b_1}{H_1}, \tag{55}$$

$$\Delta t_2 = \frac{b_2}{H(t_2)} = \frac{b_2}{H_2}. \tag{56}$$

Thus, the larger b_i, the slower is the transition in comparison with the Hubble time at t_i, H_i^{-1}. It is usual to take $b_1 \sim b_2 \sim 1$, but, as for the first transition, there is no compelling reason to assume that it necessarily occurred in a time scale of the order of H_1^{-1} in every model. For example, this transition may occur as fast as $10^{-4}H_1^{-1}$ in the ''new'' inflationary scenarios [2].

When the first stage is not a de Sitter one, that is, for $\gamma<2/3$, $\gamma\neq 0$, the constant κ and the function $G(t)$ become

$$\kappa = \frac{4}{3}, \tag{57}$$

$$G(t) = \frac{1}{\pi\sigma^3}G_1(t) + \frac{4}{|2m+1|^4}C_0, \tag{58}$$

where

$$m \equiv \frac{3}{2}\frac{(2-\gamma)}{(3\gamma-2)} < 0, \tag{59}$$

$$C_0 \equiv \frac{1}{4}\left(m-\frac{1}{2}\right)^2\left(m+\frac{1}{2}\right)^2\int_{y_2}^{y_1}B_m(y)dy + D(y_1) - D(y_2), \tag{60}$$

$$y_1 \equiv \frac{\pi|2m+1|}{b_1}, \tag{61}$$

$$y_2 \equiv \frac{\pi|2m+1|}{b_2\,\sigma}, \tag{62}$$

$$D(y) \equiv f_1(m,y)B_m(y) + f_2(m,y)B_{m+1}(y) + f_3(m,y)C_m(y) + \frac{y^4}{\pi}, \tag{63}$$

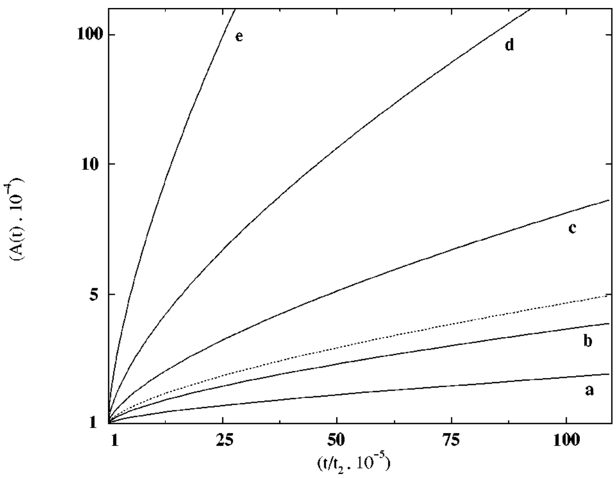

FIG. 1. The quantity $A=a/a_2$ as a function of time [solution of Eq. (41)] for $b_1=1$, $b_2=1$, $h_1=10^{-50}$, $\gamma=0.5$, and five values of t_2-t_1: 3×10^6 yr (curve a), 10^6 yr (curve b), 3×10^5 yr (curve c), 10^5 yr (curve d), and 3×10^4 yr (curve e). The dotted curve corresponds to $a(t)\propto t^{2/3}$. The solutions are insensitive to all parameters, except t_2-t_1.

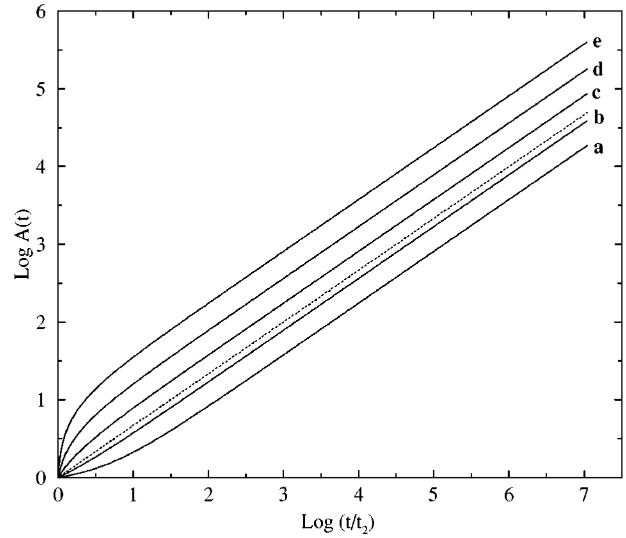

FIG. 2. The same as in Fig. 1, but using a logarithmic scale. Note that all models behave asymptotically as $t^{2/3}$. The logarithm is to base 10.

$$f_1(m,y)\equiv\frac{1}{4}\left[y^5+\frac{1}{2}\left(m+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)y^3-\frac{1}{2}\left(m-\frac{1}{2}\right)^2\left(m+\frac{1}{2}\right)y\right], \tag{64}$$

$$f_2(m,y)\equiv\frac{1}{4}\left[y^5+\frac{1}{2}\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)y^3\right], \tag{65}$$

$$f_3(m,y)\equiv-\frac{1}{2}\left(m+\frac{1}{2}\right)\left[y^4+\frac{1}{2}\left(m-\frac{1}{2}\right)^2y^2\right], \tag{66}$$

$$B_m(y)\equiv J_m^2(y)+Y_m^2(y), \tag{67}$$

$$C_m(y)\equiv J_m(y)J_{m+1}(y)+Y_m(y)Y_{m+1}(y). \tag{68}$$

J_m and Y_m are, respectively, the Bessel functions of the first and second kinds of order m, and

$$G_1(t)\equiv\frac{1}{32\sigma}\ln\tau+P(m,\sigma,b_2)\left[\frac{\pi^2}{b_2^3(-2m-3)}(\tau^{-2m-3}-1)+\frac{b_2}{512\pi^2(1-2m)}(\tau^{1-2m}-1)+\frac{1}{128\pi}f(t)\right], \tag{69}$$

$$P(m,\sigma,b_2)\equiv\left(\frac{2\,b_2\,\sigma}{\pi|2m+1|}\right)^{-2m}\Gamma^2(-m)|2m+1|, \tag{70}$$

$$f(t)\equiv P_1(m,\sigma)\tau^{-2m}\sin\theta_1+\left[\frac{4\pi}{b_2}P_2(m,\sigma)\tau^{-2m-1}-\frac{b_2}{4\pi(1-2m)}\tau^{1-2m}\right]\cos\theta_1+P_3(m,\sigma,b_2)\cos\theta_2-P_1(m,\sigma)\sin\theta_2+P_4(m,\sigma)I(t), \tag{71}$$

$$I(t)\equiv\int_{\theta_1(t)}^{\theta_2}\frac{\sin\theta}{\theta^{-2m-1}}d\theta, \tag{72}$$

$$P_1(m,\sigma)\equiv\frac{3\sigma-2m(1+2\sigma)}{(-2m)(1-2m)\sigma}, \tag{73}$$

$$P_2(m,\sigma)\equiv\frac{(\sigma-1)}{\sigma^2|2m+1|}\left[\frac{3\sigma-2m(1+2\sigma)}{(-2m)(1-2m)}+\frac{2\sigma(\sigma+1)}{(\sigma-1)}\right], \tag{74}$$

$$P_3(m,\sigma,b_2)\equiv\frac{b_2}{4\pi(1-2m)}-\frac{4\pi}{b_2}P_2(m,\sigma), \tag{75}$$

$$P_4(m,\sigma)\equiv-\frac{1}{\sigma|2m+1|}\left[\frac{4(\sigma-1)}{|2m+1|}\right]^{-2m}\left[\frac{3\sigma-2m(1+2\sigma)}{(-2m)(1-2m)}+\frac{2\sigma(\sigma+1)}{(\sigma-1)}+\frac{2\sigma^2|2m+1|}{(\sigma-1)^2}\right], \tag{76}$$

and Γ is the gamma function.

IV. NUMERICAL RESULTS AND CONCLUSIONS

We are interested in evaluating $a(t)$ after t_2; hence the initial conditions are

$$A(t_2)=1 \tag{77}$$

and

$$\dot{A}(t_2)=\frac{H_1}{\sigma^2}. \tag{78}$$

Note that, besides b_1 and b_2, there are three free parameters:

$$h_1 \equiv H_1/m_{Pl}, \tag{79}$$

σ, which we relate to the duration of the radiation era, t_2-t_1, through

$$\sigma^2=1+2H_1(t_2-t_1) \tag{80}$$

and m (or, equivalently, γ).

We have integrated Eq. (41) numerically for different values of the parameters b_1, b_2, h_1, γ, and t_2-t_1. We have taken as characteristic values $b_1=1$, $b_2=1$, $h_1=10^{-50}$, and $0\leqslant\gamma\leqslant 0.6$. In Fig. 1 we show the solutions for five values of t_2-t_1, namely, 3×10^6 yr (curve a), 10^6 yr (curve b), 3×10^5 yr (curve c), 10^5 yr (curve d), and 3×10^4 yr (curve e). The dotted curve represents the standard behavior $a(t)\propto t^{2/3}$. The time is measured in units of the time of the beginning of the matter era t_2. The important thing to notice is that the solutions are insensitive to all parameters, except t_2-t_1. We see that the lookback time for curves a and b is larger than in the standard case, thus representing older universes. The opposite happens for models described by curves c, d, and e.

We have used a logarithmic scale in Fig. 2 in order to make clear that, asymptotically, all curves behave as $t^{2/3}$. We are led to conclude that, during the matter epoch, the transformation of very long tensor perturbations into effective gravitational waves changes the expansion dynamics only near t_2. This is in contrast to what happens during the radiation era, as has been shown in Ref. [1]. Therefore, the Sahni's conjecture [12] will hold only under very restrictive conditions.

We must remark, however, that, in the model analyzed above, we have assumed a standard evolution from a radiation phase with $a(t)\propto t^{1/2}$ to a dust era with $a(t)\propto t^{2/3}$. We have not taken into account the modifications in the dynamics generated by the graviton back reaction during the radiation-dominated period [1]. A more realistic model would consider these modifications and *then* a transition to a matter-dominated era. The numerical work becomes much more involved in this case. Moreover, primordial nucleosynthesis and the data related to the anisotropy of the cosmic microwave background will place severe constraints on the relevant parameters. These questions are presently under investigation.

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